Math 561 Test #1 Conceptual Foundations

Chapter 1

* Standard form: ① Minimization ② Non-redundant equations ③ non-negative variables
* absolute values:

S.F: $\min c'x$ ← scalar

$A \in \mathbb{R}^{m \times n}$ $\quad Ax = b$ ← $\mathbb{R}^m$ ($Ax \in \mathbb{R}^n$)

$x \geq 0$

Dual: $\max y'b$ ← scaler

$y'A \leq c'$ ← $\mathbb{R}^n$

* when dealing with slack for a dual variable

$Ax + t = b \Rightarrow y'I \leq 0$

$Ax - t = b \Rightarrow y'I \geq 0$

* Weak Duality: If $\hat{x}$ is feasible in (P) and $\hat{y}$ feasible in (D), then $c'\hat{x} \geq \hat{y}'b$

Chapter 2

Production problem: m resources, in quantities b_i, $i=1,2,\dots,m$,
n production activities, profit c_j, $j=1,2,\dots,n$.
Each unit of activity consumes a_{ij} units of resource i

$$\max \; c'x \qquad \text{dual} \quad c=(c_1,c_2,\dots,c_n)'$$
$$Ax \le b \qquad y \quad b=(b_1,b_2,\dots,b_m)'$$
$$x \ge 0$$

dual $\min \; y'b$
$$y'A \ge c'$$
$$y \ge 0$$

Norm minimization:

$\|x\|_\infty$: $\min t$
$$t - x_i \ge 0, \quad t + x_i \ge 0, \quad Ax = b$$

$\|x\|_1$: similar idea of getting the magnitude
$$\min \sum_{i=1}^{n} t_i$$
$$t_i + x_i \ge 0, \quad t_i - x_i \ge 0, \quad Ax = b$$

Network Flow: Nodes N, arcs A

x_e^k := amt of flow of commodity k on arc e

u_e := flow u.b on arc e.

c_e^k := cost of per-unit flow of commodity k on arc e.

b_v^k := total supply of commodity k on arc

LP Ch3 Definitions

$\min c'x$
$Ax = b$
$x \geq 0$

$\boxed{3.1}$ Basic Feasible Solutions and Extreme points: $A \in \mathbb{R}^{m \times n}$

Basic solution:

$\bar{x}_n = 0 \quad \in \mathbb{R}^{n-m}$

$x_\beta = A_\beta^{-1} b$

Feasible:

$\bar{x}_\beta = A_\beta^{-1} b \geq \vec{0}$

Feasible region: solution set of

$x_\beta + A_\beta^{-1} A_n x_n = A_\beta^{-1} b \quad$ w/ $x_\beta, x_n \geq 0$

Geometry

☆ Project feasible region into the space of non-basic variables:

$(A_\beta^{-1} A_n) x_n \leq A_\beta^{-1} b \quad$ w/ $x_n \geq 0 \quad$ (x_β is like a slack variable)

☆ [Thm 3.2: Every BFS of SF (P) is an extreme point of its region.
Thm 3.3: Every extreme point of the feasible region of SF (P) is a basic solution.

$\boxed{3.2}$ Basic Feasible Directions

FD wrt FS $\hat{x}$ is a $\hat{z} \in \mathbb{R}^n$, $\hat{z} \neq 0$, s.t. $\hat{x} + \varepsilon \hat{z} \in S$ for $\varepsilon > 0$

Basic Direction:

$\hat{z}_n = e_j$

$\hat{z}_\beta = -A_\beta^{-1} A_{n_j}$

BFD:

$A_\beta^{-1} b - \varepsilon A_\beta^{-1} A_{n_j} \geq 0$

Thm 3.5: Let $\bar{A} = A_\beta^{-1} A_{n_j}$, a BD is a BFD wrt FS $\bar{x}$ iff

$\bar{x}_{\beta_i} > 0 \quad \forall i$ s.t. $\bar{a}_{i,n_j} > 0$.

☆ for the negative components of $\hat{z}_\beta$, $\hat{x}_\beta$ component must be positive.

$\boxed{3.3}$ BFR & Extreme Rays

☆ Thm 3.6: BD $\hat{z}$ is a ray of FR (P) iff $\bar{A}_{n_j} \leq 0$.

☆ Same as $\hat{z} \geq 0$.

Extreme ray: cannot write $\hat{z} = z^1 + z^2$, w/ $z^1 \neq \mu z^2$ being rays of S and $\mu \neq 0$

Thm 3.7: Every BFR is an extreme ray

Thm 3.8: Every extreme ray of FR (P) is a positive multiple of a BFR.

Chapter 4

$\boxed{4.1}$ A sufficient optimality criterion

The dual solution of (D) associated w/ B is : $\bar{y}' = c' A_B^{-1}$

Reduced costs: $\bar{c}' := c' - c'_B A_B^{-1} A = c' - \bar{y}' A$

Lemma 4.2: The dual solution of (D) is feasible iff $\bar{c}_n' \geq 0$

pf. $c_n' - \bar{y}' A_n \geq 0 \Rightarrow y' A_n \leq c_n'$. By def $y' = c_B' A_B^{-1} \Rightarrow y' A_B = c_B'$

$y' A = y' [A_B \; A_n] \leq [c_B' \; c_n']$

Lemma 4.1: Given basis B, (P) & (D) solutions have equal objective value.

p.f. follows from definitions

Lemma 4.3: If B is a feasible basis and dual feasible basis, then primal solution $\bar{x}$ and dual solution $\bar{y}$ are optimal.

p.f: objective equality shown in 4.2, follows from weak duality.

4.2 No Worries Simplex Algorithm

When we have not reached sufficient optimality conditions ($\bar{c}_n \geq 0$)

- Choose n_j s.t. $\bar{c}_{n_j} < 0$ $\qquad \bar{c}_{n_j} = c_{n_j} - c_\beta' A_\beta^{-1} A_{n_j}$
- Consider solutions that increase the value of x_{n_j}, up from $\bar{x}_{n_j} = 0$

Take basic direction $\hat{z} \in \mathbb{R}^n$:

$\bar{z}_n := \hat{e}_j \quad \in \mathbb{R}^{n-m}$

$\bar{z}_\beta := -A_\beta^{-1} A_{n_j} = -\bar{A}_{n_j} \quad \in \mathbb{R}^m$

Consider solutions $\bar{x} + \lambda \bar{z}$ w/ $\lambda > 0$

Motivation

$$c'(\bar{x} + \lambda \bar{z}) - c'\bar{x} = c' \begin{pmatrix} x_\beta + \lambda A_\beta^{-1} A_{n_j} \\ x_n + \lambda \hat{e}_j \end{pmatrix} - c' \begin{pmatrix} x_\beta \\ x_n \end{pmatrix}$$

$$= \lambda(-c_\beta' A_\beta^{-1} A_{n_j}) + \lambda c_n' \hat{e}_j = \lambda (c_{n_j} - c_\beta' A_\beta^{-1} A_{n_j}) \bar{c}_{n_j}$$

$< 0 \therefore$ obj $\downarrow$

$$A(\bar{x} + \lambda \bar{z}) = A\bar{x} - \lambda A \begin{pmatrix} -A_\beta^{-1} A_{n_j} \\ e_j \end{pmatrix} \quad (-A_\beta A_\beta^{-1} A_{n_j} + A_{n_j}) = 0$$

Maximum step: choose λ s.t.

$\bar{x}_\beta + \lambda \bar{z}_\beta = \bar{x}_\beta - \lambda \bar{A}_{n_j} \geq 0 \quad \longrightarrow$ if $\bar{A}_{n_j} \leq 0$, there is no limit on $\lambda \therefore$ unbounded

If $\bar{A}_{n_j}$ not ≤ 0, we only look at $\bar{a}_{i,n_j} > 0$

enforce: $\lambda \leq \dfrac{\bar{x}_{\beta_i}}{\bar{a}_{i,n_j}}$ $\quad \Rightarrow \bar{\lambda} = \min\limits_{i, \bar{a}_{i,n_j} > 0} \left\{ \dfrac{\bar{x}_{\beta_i}}{\bar{a}_{i,n_j}} \right\}$

☆ Non degeneracy hypothesis: For every feasible basis β, we have $\bar{x}_{\beta_i} > 0$. $\Rightarrow \bar{\lambda} > 0$

From our construction of lambda, one former basic index has become 0. This is

$i^* = \operatorname{argmin}\limits_{i, \bar{a}_{i,n_j} > 0} \left\{ \dfrac{\bar{x}_{\beta_i}}{\bar{a}_{i,n_j}} \right\}$ $\therefore$ In our new basic solution, we replace $x_{\beta_{i^*}}$ with x_{n_j}.

☆ $\bar{x} + \bar{\lambda} \bar{z}$ is the basic solution determined by $\bar{\beta}, \bar{n}$.

Worry Free Alg

0. Start w/ β, n.
1. Compute (P) and (D) solutions $\bar{x}, \bar{y}$. If $\bar{c}_n \geq 0 \Rightarrow$ STOP (✓)
2. Otherwise, choose n_j s.t. $\bar{c}_{n_j} < 0$
3. If $\bar{A}_{n_j} \leq 0 \Rightarrow$ STOP (unbounded)
4. Select $i^* = \operatorname{argmin}\limits_{i, \bar{a}_{i,n_j} > 0} \left\{ \dfrac{\bar{x}_{\beta_i}}{\bar{a}_{i,n_j}} \right\}$, β_i leaves, n_j joins
5. GOTO 1

Intuition

$\bar{c}_n - y'A_n \geq 0 \Rightarrow y'A_n \leq c_n$, $y'A_\beta = c_\beta$.

- $\bar{c}_n \geq 0 \Rightarrow$ (D) is feasible $\Rightarrow$ optimal
- want to decrease obj value
- no limit to direction, any ⊕ λ won't violate constraints
- when we choose $\bar{\lambda}$, some basic index $= 0$.

4.3

Lemma 4.11: The ε-perturbed problem satisfies the non degeneracy hypothesis

Thm 4.12: Let β° be a basis feasible for (P). Then WFS Alg applied to $P_\varepsilon(A_{\beta^\circ})$, starting from β°, correctly demonstrates that (P) is unbounded or finds an optimal basic partition for (P).

4.4

Pick any basic partition $\bar{\beta}, \bar{\eta}$. If $A_{\bar{\beta}}^{-1} b$ is not ≥ 0.

Consider Phase one problem, $A_{n+1} = -A_{\bar{\beta}} \mathbf{1}$

$$\min \quad x_{n+1}$$

$$Ax + A_{n+1} x_{n+1} = b$$

$$x \geq 0, \; x_{n+1} \geq 0$$

Starting basis: Choose i^* so that $x_{\bar{\beta}_i}$ is most $\ominus$

$\beta := (\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_{i^*-1}, n+1, \bar{\beta}_{i^*+1}, \dots \bar{\beta}_m)$

$\eta := (\bar{\eta}_1, \dots, \bar{\eta}_{n-m}, \bar{\beta}_{i^*})$

Not ignoring degeneracy

"Early arrival": In ε-perturbed, x_{n+1} decrease to a homogenous polynomial (leading term is 0). Then let x_{n+1} leave basis and terminate.

Thm 4.14: If standard form (P) has a feasible solution, then it has a basic feasible sol.

"Be patient" Solve P_ε in full.

4.5 The Simplex Algorithm

1. Apply ε-perturbation to phase one problem
2. Solve phase one w/ WFS, giving preference for n+1 leaving the basis.
3. Starting from feasible basis, apply new perturbation
4. Solve the problem using WFS.

Chapter 5 - Duality

Weak duality: If $\hat{x}$ is feasible in (P) and $\hat{y}$ is feasible in (D), then $c'\hat{x} \geq \hat{y}'b$

Weak Optimal Basis: If β is a feasible basis and $\bar{c}_N \geq 0$, then the primal solution $\bar{x}$ and the dual solution $\bar{y}$ associated associated w/ β are optimal.

THM 5.1 (Strong Optimal Basis Thm) p.f uses simplex alg.
If (P) has a feasible solution and (P) is not unbounded, then there exists basis β such that the associated basic solution $\bar{x}$ and the associated dual solution $\bar{y}$ are optimal. Moreover, $c'x = y'b$.

THM 5.2 (Strong Duality Thm)
If (P) has a feasible solution, and (P) is not unbounded, then there exists feasible solutions $\hat{x}$ for (P) and $\hat{y}$ for (D) that are optimal. Moreover, $c'x = y'b$.

Complementary Slackness 5.2

DEF
Wrt (P) and (D), solutions $\hat{x}$ and $\hat{y}$ are complementary if $\begin{cases} (c_j - \hat{y}'A_{\cdot j})\hat{x}_j = 0 & \text{for } j=1,\dots,n \\ \hat{y}_i(A_{i\cdot}\hat{x} - b_i) = 0 & \text{for } i=1,\dots,m \end{cases}$

Thm 5.3: If β is a basis, then the primal basic solution $\hat{x}$ and the dual solution $\hat{y}$ are complementary.

Thm 5.4: If $\hat{x}$ and $\hat{y}$ are complementary w.r.t (P) and (D), then $c'\hat{x} = \hat{y}'b$. p.f: follows from 5.3

COR 5.5: (Weak comp. slackn. thm) - If $\hat{x}$ and $\hat{y}$ are feasible and complementary wrt (P) & (D), then $\hat{x}$ and $\hat{y}$ are optimal. p.f: From 5.4 & Weak duality.

Thm 5.6: If $\hat{x}$ and $\hat{y}$ are optimal for (P) and (D), then $\hat{x}$ and $\hat{y}$ are complementary.

Strong Complementary Slackness

Duality for general Linear-Optimization Problems

	min	max	
const	$\geq$	≥ 0	var
	$\leq$	≤ 0	
	$=$	unconstrained	
var	≥ 0	$\leq$	const
	≤ 0	$\geq$	
	unres	$=$	

$\boxed{5.4}$ Theorems of the Alternative

Farkas Lemma: Exactly one system has a solution.

(I) $Ax = b$, $x \geq 0$ (II) $y'b > 0$, $y'A \leq 0$

Thm 5.11.

(I) $Ax \geq b$ (II) $y'b > 0$, $y'A = 0$, $y \geq 0$

Chapter 6: Sensitivity Analysis

$\boxed{1}$ RHS Changes 1. Local analysis Let $h^i = A_\beta^{-1} e_i$ so $[h^1, h^2, \dots, h^m] = A_\beta^{-1}$

(P_i) $\min c'x$
$Ax = b + \Delta_i e_i$
$x \geq 0$

$A_\beta^{-1}(b + \Delta_i e_i) \geq 0 \Rightarrow x_\beta + \Delta_i h^i \geq 0$

Δ_i must be in interval $\begin{cases} L_i = \max\limits_{k: h^i_k > 0} \{-\bar{x}_{\beta k} / h^i_k\} \\ U_i = \min\limits_{k: h^i_k < 0} \{-\bar{x}_{\beta k} / h^i_k\} \end{cases}$

1.2 Global analysis

Thm 6.1: The domain of f is a convex set

$f(b) = \min c'x$
$Ax = b$ (P_b)
$x \geq 0$

Def: f is a convex function on its domain S if:

$f(\lambda u^1 + (1-\lambda)u^2) \leq \lambda f(u^1) + (1-\lambda) f(u^2)$ $\forall u^1, u^2 \in S$, $0 < \lambda < 1$

Def: f is an affine function if it has the form

$f(u_1, \dots, u_m) = a_0 + \sum_{i=1}^{m} a_i u_i$.

Def: A function $f: \mathbb{R}^m \to \mathbb{R}$ having a convex set as it's domain is a convex piecewise-linear function if, on its domain, it is the pointwise maximum of a finite number of affine functions.

Thm 6.2: If f is a convex piecewise linear function, then it is a convex function.

Thm 6.3: f is a convex piecewise-linear function on it's domain.

6.2 : Objective changes

$$g(c) := \begin{array}{l} \min c'x \\ Ax = b \\ x \geq 0 \end{array}$$

Local Analysis

C is the solution set of c s.t $c_n' - c_B' A_B^{-1} A_n \geq 0$

Global :

Domain of g is convex

Def of concave : $g(\lambda u^1 + (1-\lambda) u^2) \geq \lambda g(u^1) + (1-\lambda) g(u^2)$

Thm 6.5 : g is concave piecewise linear on its domain

Chapter 7: Large Scale Linear Optimization

Motivation: Might have very efficient way to solve a linear optimization problem if certain "complicating" constraints weren't getting in the way.

Thm 7.1 (The Representation Theorem)

Let

$$(P)\quad \begin{aligned} \min\ & c'x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Suppose (P) has a nonempty feasible region. Let $\mathcal{X} = \{\hat{x}^j : j \in J\}$ be the set of basic feasible solutions of (P), and let $\mathcal{Z} = \{\hat{z}^k : k \in K\}$ be the set of B.F rays of (P).

Then the feasible region of (P) is equal to:

$$\left\{ \sum_{j \in J} \lambda_j \hat{x}^j + \sum_{k \in K} \mu_k \hat{z}^k : \sum_{j \in J} \lambda_j = 1 ;\ \lambda_j \geq 0,\ j \in J ;\ \mu_k \geq 0,\ k \in K \right\}$$

Corollary 7.2 (The Decomposition Theorem)

Let

$$(Q)\quad \begin{aligned} z := \min\ & c'x \\ & Ex \geq h \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Let $S := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$, let $\mathcal{X} := \{\hat{x}^j : j \in J\}$ and $\mathcal{Z} = \{\hat{z}^k : k \in K\}$ be the set of B.F solutions and rays of S.

Then (Q) is equivalent to the <u>main problem</u>

$$(M)\quad \begin{aligned} \min\ & \sum_{j \in J} (c'\hat{x}^j)\lambda_j + \sum_{k \in K} (c'\hat{z}^k)\mu_k \\ & \sum_{j \in J} (E\hat{x}^j)\lambda_j + \sum_{k \in K} (E\hat{z}^k)\mu_k \geq h \\ & \sum_{j \in J} \lambda_j = 1 \quad ;\ \lambda_j \geq 0,\ j \in J ;\ \mu_k \geq 0,\ k \in K \end{aligned}$$

Solution of Main Problem w/ Simplex Algorithm:

* (M) is too big to write out explicitly — but can maintain BFS

$(\bar{M})$ min $\sum c'\hat{x}^j \lambda_j + \sum c'\hat{z}^k \mu_k$ <u>duals</u>

$\sum E\hat{x}^j\lambda_j + \sum E\hat{z}^k \mu_k - Is = h$ $\bar{q} \geq 0$

$\sum \lambda_j = 1$ $\lambda_j \geq 0$, $\mu_k \geq 0$, $s \geq \vec{0}$ $\bar{\sigma}$ unrestricted

<u>Entering variable</u>: Only step where simplex is sensitive to size is checking reduced cost.

s_i: if $q_i \leq 0$

λ_j: reduced cost $= c'x^j - \bar{q}'E\hat{x}^j - \bar{\sigma} = -\bar{\sigma} + (c' - \bar{q}'E)\hat{x}^j$

(SUB) $-\bar{\sigma} + \min (c' - q'E)x$

$Ax = b$, $x \geq 0$

* If optimal obj of (SUB) is negative, it has an $\hat{x}^j$ whose associated λ_j can enter basis. otherwise, proof that no λ_j is eligible.

μ_k: If (SUB) is unbounded

<u>Leaving variable</u>:

if entering: λ_j μ_k s_i

Ratio test needs: $B^{-1}\begin{pmatrix} h \\ 1 \end{pmatrix}$ and $B^{-1}\begin{pmatrix} E\hat{x}^j \\ 1 \end{pmatrix}$, $B^{-1}\begin{pmatrix} E\hat{z}^k \\ 0 \end{pmatrix}$, $B^{-1}\begin{pmatrix} -e_i \\ 0 \end{pmatrix}$

Convergence of Decomposition Algorithms:

* We want a good lower bound on z, easy to solve systems w/ $Ax=b$

Lagrangian bounds

$$(L\hat{y}) \quad v(\hat{y}) = \hat{y}'h + \min (c' - \hat{y}'E)x \qquad Ax = b \; ; \; x \geq 0$$

Thm 7.3: $v(\hat{y}) \leq z$, $\forall \hat{y}$ in the domain of v.

Thm 7.4: Suppose x^* is optimal for (Q), and suppose $\hat{y}$ and $\hat{\pi}$ are optimal for the dual of (Q). Then x^* is optimal for $(L\hat{y})$, $\hat{\pi}$ is optimal for the dual of $(L\hat{y})$, $\hat{y}$ is a maximizer of $v(y)$ over $y \geq 0$, and the max value of $v(y)$ over $y \geq 0$ is z.

$$(D_Q) \quad \max\ y'h + \pi'b \qquad y'E + \pi'A \leq c' \qquad y \geq 0$$

$$(D_{L\hat{y}}) \quad \hat{y}'h + \max\ \pi'b \qquad \pi'A \leq c' - \hat{y}'E$$

Thm 7.5: Suppose $\hat{y}$ is a maximizer of $v(y)$ over $y \geq 0$ and suppose $\hat{\pi}$ is optimal for the dual of $(L\hat{y})$. Then $\hat{y}$ and $\hat{\pi}$ are optimal for the dual of (Q) and the optimal value of (Q) is $v(\hat{y})$.

Solving the Lagrangian Dual: Thm 7.3 gives good LB on z if we have good $\hat{y}$.

:: Maximize $v(\hat{y})$

Thm 7.6: Suppose we fix $\hat{y}$ and solve for $v(\hat{y})$. Let $\hat{x}$ be the solution of $(L\hat{y})$. Let $\hat{\gamma} = h - E\hat{x}$. Then $v(\tilde{y}) \leq v(\hat{y}) + (\tilde{y} - \hat{y})\hat{\gamma}$ $\forall \tilde{y}$ in domain of v.

Projected Subgradient Opt Alg:

0. Non-negative $\hat{y}^1 \in \mathbb{R}^n$. $k=1$
1. Solve $(L\hat{y}^k)$ to get $\hat{x}^k$
2. $\hat{\gamma}^k = h - E\hat{x}^k$
3. $\hat{y}^{k+1} = \mathrm{Proj}_{\mathbb{R}^n_+}(\hat{y}^k + \lambda_k \hat{\gamma}^k)$
4. $k \leftarrow k+1$, GOTO 1.

Convergence

λ_i: "Square summable but not summable"

λ_i: "$\lim_{i\to\infty} \lambda_i = 0$ and $\sum \lambda_i = +\infty$"

Chapter 8: Integer Linear Optimization

8.1: Integrality for free

Network Flow Model

Network G:

Nodes N - set

arcs A - each arc e has tail and head in N

☆ Single commodoty allowed to flow along each arc : x_e

- non negative, should not exceed $u_e :=$ flow UB
- each arc has a cost c_e

Assume each node has supply b_v

- Flow is conservative if net flow out of v minus net flow into v is equal to net supply.

Single commodoty min cost network flow:

$$\min \sum_{e \in A} c_e x_e$$

$$\sum_{t(e)=v} x_e - \sum_{h(e)=v} x_e = b_v \quad \forall v \in N$$

$$0 \le x_e \le u_e, \quad \forall e \in A$$

Matrix formulation:

$$a_{ve} := \begin{cases} 1 & \text{if } t(e)=v \\ -1 & \text{if } h(e)=v \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} \min\ & c'x \\ & Ax = b \\ & x \le u \\ & x \ge 0 \end{aligned}$$

8.2 : Modeling Techniques

Disjunctions

$-12 \le x \le 2$ or $5 \le x \le 20$

* We can introduce binary variable $y \in \{0, 1\}$, model disjunction as

$$x \le 2 + M_1 y \qquad\qquad x \le 2 + 18y$$
$$\Rightarrow$$
$$x + M_2(1-y) \ge 5 \qquad\qquad x + 17(1-y) \ge 5$$

practice

$-30 \le x \le -15$ or $80 \le x \le 95$

$$x \le -15 + M_1 y \qquad y=1:\ M_1 = 110 \qquad x \le -15 + 110y$$
$$\Rightarrow$$
$$x + M_2(1-y) \ge 80 \qquad y=0:\ M_2 = 110 \qquad x + 110(1-y) \ge 80$$

Forcing Constraints

Uncapacitated facility location problem: n customers, m facilities

f_i: fixed cost for operating facility

c_{ij}: cost of satisfying customer j's demands from facility i

y_i: indicator var for operating facility i

x_{ij}: fraction of customer j's demand satisfied by facility i

Formulation:

$$\min \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad \text{for } j=1,\dots,n$$

$$(s) \quad -y_i + x_{ij} \le 0 \quad \text{for } i=1,\dots,m,\ j=1,\dots,n$$

$$y_i \in \{0,1\} \quad \text{for } i=1,\dots,m$$

$$x_{ij} \ge 0 \quad \text{for } i=1,\dots,m,\ j=1,\dots,n$$

other forcing constraint: $-ny_i + \sum_{j=1}^{n} x_{ij} \le 0$ for $i=1,\dots,m$ (w)

Branch and Bound

$$(D_{\mathcal{I}}) \quad \begin{aligned} z = \max\ & y'b \\ & y'A \le c' \\ & y \in \mathbb{R}^m, \quad y_i \text{ integer for } i \in I \end{aligned}$$

$$(P) \quad \begin{aligned} \min\ & c'x \\ & Ax = b \\ & x \ge 0 \end{aligned} \quad \text{"dual of cont. relaxation"}$$

Key invariant: Every feasible solution of the original problem $(D_{\mathcal{I}})$ with greater obj than LB is feasible for a problem on the list.

☆ Stop when list is empty, LB = optimal value

$\mathcal{L}$: integer optimization problems w/ general form of $(D_{\mathcal{I}})$

Step

1. Remove problem $(\bar{D}_{\mathcal{I}})$ from list and solve its continuous relaxation $(\bar{D})$
   Let $\bar{y}$ be it's optimal solution

2. If $\bar{y}'b \le LB$, then no feasible solution can have greater obj val than LB.

   If $\bar{y}'b > LB$:

   If y is integer: Update LB and $\bar{y}_{LB}$

   If y_i is not integer $\forall i \in \mathcal{I}$, then select some $i \in \mathcal{I}$

   - Down branch: add $y_i \le \lfloor \bar{y}_i \rfloor$ to list
   - Up branch: add $y_i \ge \lceil \bar{y}_i \rceil \Rightarrow -y_i \le -\lceil y_i \rceil$ to list

3. Thm 8.17: Suppose (P) is feasible. Then @ termination, we have $LB = -\infty$ if $(D_{\mathcal{I}})$ is infeasible or with $\bar{y}_{LB}$ being an optimal solution of $(D_{\mathcal{I}})$

Solving continuous relaxations

down branch: $y_i \le \lfloor \bar{y}_i \rfloor$ dualizes to new x_{down} w/ cost $\lfloor \bar{y}_i \rfloor$ $A_{down} = e_i$
reduced cost: $\lfloor \bar{y}_i \rfloor - \bar{y}_i e_i = \lfloor \bar{y}_i \rfloor - \bar{y}_i \le 0$ ∴ fit to enter

up branch: $y_i \le -\lceil \bar{y}_i \rceil$ dualizes to x_{up} w/ cost $-\lceil \bar{y}_i \rceil$ and $A_{down} = -e_i$
reduced cost $-\lceil \bar{y}_i \rceil + \bar{y}_i \le 0$

Partially solving: If (P) falls below LB while solving, we can terminate

Selecting problems from list

hybrid
- LIFO: diving - can be good choice
- FIFO: BAD
- Best Bound: Choose problem w/ obj val = UB - solve cont before putting on list.